SPREADING OF A TWISTED STREAM IN AN INFINITE SPACE FLOODED BY THE SAME FLUID

(RASPROSTRANENIE ZAKRUCHENNOI STRUI V BEZGRANICHNOM PROSTRANSTVE, ZATOPLENNOM TOI ZHE ZHIDKOST'IU)

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The spreading of a twisted stream flowing through a round orifice into an infinite space flooded by the same fluid at rest is of interest in many technical problems.

The formulation of this problem and its first solution are due to Loitsianskii [1], who had derived an asymptotic solution for the boundary layer of a lightly twisted stream, valid at a great distance from the orifice. He had found the two first terms of the expansion which, however, does not permit to analyze the effect of the twist velocity on the stream axial velocity, nor to determine the area where the reverse stream originates.

In this paper the third and fourth terms of the asymptotic expansion are derived in their final form. This makes possible the analysis of streams with more pronounced twist, as well as the effect of twisting on the axial velocity profile.

1. Fundamental equations. In the case of axial symmetry the equations of a viscous incompressible fluid boundary layer of a twisted stream have, in a cylindrical coordinate system, the following form:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right),\qquad \frac{w^2}{r}=\frac{1}{\rho}\frac{\partial p}{\partial r}\tag{1.1}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial r}+\frac{vw}{r}=\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{w}{r^2}\right),\qquad \frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial r}(rv)=0$$

Here, u, v, w are respectively the axial, radial and transversal components of the velocity vector, x is the axial distance from the stream source, and r is the distance from the stream axis.

Applying the theorem of change of momentum and of the moment of momentum, we obtain

$$\int_0^\infty r(p+\rho u^2)\,dr=\frac{K_0}{2\pi},\qquad \int_0^\infty r^2uw\,dr=\frac{L_0}{2\pi\rho}\tag{1.2}$$

Here K_0 and L_0 are constants which characterize the initial impulse and the kinetic moment of the stream.

2. Asymptotic expansion of velocities and pressure. From the last Eq. of (1.1) it follows that the axial and radial components of the velocity vector may be expressed by one function $\psi(x, r)$, if we assume

$$u=\frac{1}{r}\frac{\partial\psi}{\partial r},\qquad v=-\frac{1}{r}\frac{\partial\psi}{\partial x}\tag{2.1}$$

We introduce new independent variables

$$\xi = x, \qquad \eta = \frac{r}{x\sqrt{\nu}} \tag{2.2}$$

Following Loitsianskii's method we shall seek function $\psi(x, r)$ in the form of expansion

$$\psi = \nu\left(\bar{a}x + a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right) \tag{2.3}$$

Here, $\bar{a}, a_0, a_1, \ldots$ are the unknown functions η. From (2.1) we derive the expressions for velocity components u and v

$$u = \frac{\bar{a}'}{\eta}\frac{1}{x} + \frac{a_0'}{\eta}\frac{1}{x^2} + \frac{a_1'}{\eta}\frac{1}{x^3} + \frac{a_2'}{\eta}\frac{1}{x^4} + \cdots \tag{2.4}$$

$$v = \frac{\sqrt{\nu}}{x}\left[\bar{a}' - \frac{\bar{a}}{\eta} + a_0'\frac{1}{x} + \left(a_1' + \frac{a_1}{\eta}\right)\frac{1}{x^2} + \left(a_2' + 2\frac{a_2}{\eta}\right)\cdot\frac{1}{x^3} + \cdots\right]$$

Here, as in the following, a prime denotes differentiation with respect to η. The notation x will be retained for ξ. The stream twist velocity w, and the pressure will be expressed in the form of series

$$w = \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \frac{b_4}{x^4} + \cdots, \qquad \frac{p}{\rho} = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \frac{c_4}{x^4} + \cdots \tag{2.5}$$

where $b_1, b_2, \ldots, c_1, c_2, \ldots$ represent the unknown functions of η. Substituting expansions (2.4) and (2.5) into the first three Eqs. of (1.1) and equating the coefficients of terms of equal powers of x, we obtain for determination of the unknown functions $\bar{a}, a_0, a_1, \ldots; b_1, b_2, \ldots; c_1, c_2, \ldots$ a system of ordinary differential equations, as follows

$$c_1 + \eta c_1' = 0 \tag{2.6}$$

$$\left(\frac{\bar{a}'}{\eta}\right)'' + \frac{1+\bar{a}}{\eta}\left(\frac{\bar{a}'}{\eta}\right)' + \left(\frac{\bar{a}'}{\eta}\right)^2 + 2c_2 + \eta c_2' = 0 \tag{2.7}$$

$$\left(\frac{a_0'}{\eta}\right)'' + \frac{1+\bar{a}}{\eta}\left(\frac{a_0'}{\eta}\right)' + \frac{3\bar{a}'}{\eta}\frac{a_0'}{\eta} + 3c_3 + \eta c_3' = 0 \tag{2.8}$$

$$\left(\frac{a_1'}{\eta}\right)'' + \frac{1+\bar{a}}{\eta}\left(\frac{a_1'}{\eta}\right)' + \frac{4\bar{a}'}{\eta}\frac{a_1'}{\eta} - \frac{1}{\eta}\left(\frac{\bar{a}'}{\eta}\right)' a_1 + \frac{2(a_0')^2}{\eta^2} + 4c_4 + \eta c_4' = 0 \tag{2.9}$$

$$\left(\frac{a_2'}{\eta}\right)'' + \frac{1+\bar{a}}{\eta}\left(\frac{a_2'}{\eta}\right)' + \frac{5\bar{a}'}{\eta}\frac{a_2'}{\eta} - 2\left(\frac{\bar{a}'}{\eta}\right)'\frac{a_2}{\eta} + 5\frac{a_0'}{\eta}\frac{a_1'}{\eta} - \frac{a_1}{\eta}\left(\frac{a_0'}{\eta}\right)' + 5c_5 + \eta c_5' = 0 \tag{2.10}$$

$$\eta c_1' = 0, \qquad \eta c_2' = b_1^2, \qquad \eta c_3' = 2b_1b_2, \qquad \eta c_4' = b_2^2 + 2b_1b_3$$

$$\eta c_5' = 2b_1b_4 + 2b_2b_3 \tag{2.11}$$

$$b_1'' + \frac{1+\bar{a}}{\eta}b_1' - \frac{1-\eta}{\eta^2}b_1 = 0, \qquad b_2'' + \frac{1+\bar{a}}{\eta}b_2' - \frac{1-\bar{a}-\eta\bar{a}'}{\eta^2}b_2 = 0 \tag{2.12}$$

$$b_3'' + \frac{1+\bar{a}}{\eta}b_3' - \frac{1-\bar{a}-2\eta\bar{a}'}{\eta^2}b_3 + \frac{a_0'b_2}{\eta} - \frac{a_1b_1'}{\eta} - \frac{a_1b_1}{\eta^2} = 0 \tag{2.13}$$

Because at the stream axis the velocity component $v = 0$ when $\eta = 0$, and $u(x, 0)$ must have a finite value, we have from (2.4)

$$\bar{a} = a_0 = a_1 = \ldots = 0 \quad \text{for} \quad \eta = 0, \quad \bar{a}' = a_0' = a_1' = \ldots = 0 \quad \text{for} \quad \eta = 0 \tag{2.14}$$

With increasing distance from the stream axis, u and v must tend to zero, therefore it follows from (2.4) that $\bar{a}(\infty)$, $a_0(\infty)$, $a_1(\infty)$ are bounded.

For the velocity of twist w we have the following obvious boundary conditions

$$b_1(0) = b_2(0) = \ldots = 0, \qquad b_1(\infty) = b_2(\infty) = \ldots = 0 \tag{2.15}$$

We assume that in Formula (2.6)

$$c_1(\infty) = c_2(\infty) = \ldots = 0 \tag{2.16}$$

Substituting expansions (2.4) and (2.5) into integrals of (1.2), and equating coefficients of terms with equal powers of x, we obtain a system of integral conditions

$$\int_0^\infty \eta c_1 d\eta = 0, \quad \int_0^\infty \left(\eta c_2 + \frac{\bar{a}'^2}{\eta}\right) d\eta = \frac{K_0}{2\pi\eta}, \quad \int_0^\infty \left(\eta c_3 + \frac{2\bar{a}' a_0}{\eta}\right) d\eta = 0$$

$$\int_0^\infty \left(\eta c_4 + \frac{a_0'^2 + 2\bar{a} a_1}{\eta}\right) d\eta = 0, \qquad \int_0^\infty \eta \bar{a}' b_1 \, d\eta = 0 \tag{2.17}$$

$$\int_0^\infty \eta\,(\bar{a}' b_2 + a_0' b_1)\, d\eta = \frac{L_0}{2\pi\mu\sqrt{\nu}}, \qquad \int_0^\infty \eta\,(\bar{a}' b_3 + a_0' b_2 + a_1' b_1)\, d\eta = 0$$

3. Solution of the problem of a lightly twisted stream. Third approximation. It may be assumed, when considering an area far away from the nozzle end, that the axial velocity component there is $u > 0$.

Because for large values of variable x the sign of u is determined by the first term of expansion (2.4), $\bar{a}'$ must be positive, hence by virtue of the fifth of the integral conditions (2.17), it follows that $b_1 \equiv 0$, as it is a constant sign parameter. With this, the first of Eqs. (2.16) is satisfied identically. The first of Eqs. (2.11) together with Eq. (2.2) yield $c_1 = 0$. The second and third of Eqs. (2.11) with their respective boundary conditions (2.16) yield $c_2 = c_3 = 0$.

This simplifies considerably Eqs. (2.8) and (2.9), which can now be integrated in their final form, as well as the homogeneous linear Eqs. (2.12).

The following expressions were derived by Loitsianskii [1] :

$$\bar{a}(\eta) = \frac{\alpha^2\eta^2}{1 + {}^1/_4\,\alpha^2\eta^2}, \qquad a_0(\eta) = -\beta\,\frac{{}^1/_4\,\alpha^2\eta^2\,(1 - {}^1/_4\,\alpha^2\eta^2)}{(1 + {}^1/_4\,\alpha^2\eta^2)^2} \tag{3.1}$$

$$b_2(\eta) = \gamma\,\frac{\alpha\eta}{(1 + {}^1/_4\,\alpha^2\eta^2)^2}, \qquad c_4(\eta) = -\frac{2}{3}\gamma^2\,\frac{1}{(1 + {}^1/_4\,\alpha^2\eta^2)^3} \tag{3.2}$$

Here α, β, γ are constants of integration. With the aid of the second and sixth of Eqs. (2.17) constants α and γ may be expressed in terms of the stream impulse K_0 and of moment of momentum L_0.

$$\alpha = \sqrt{\frac{3K_0}{16\pi\mu}}, \qquad \gamma = \frac{3\sqrt{3}}{64\pi\sqrt{\pi}}\,\frac{L_0\sqrt{\rho K_0}}{\mu^2} \tag{3.3}$$

We proceed now with the integration of Eq. (2.9) which defines the third term of expansion (2.4) of the velocity components u and v.

Substituting into (2.9) the values of variables $\bar{a}(\eta)$, $a_0(\eta)$ and $c_4(\eta)$ from Eqs. (3.1) and (3.2), we obtain for a_1 a nonhomogeneous linear equation of the third order

$$\left(\frac{a_1'}{\eta}\right)'' + \frac{1 + {}^5/_4\,\alpha^2\eta^2}{\eta\,(1 + {}^1/_4\,\alpha^2\eta^2)}\left(\frac{a_1'}{\eta}\right)' + \frac{8\alpha^2}{(1 + {}^1/_4\,\alpha^2\eta^2)^2}\,\frac{a_1}{\eta} +$$
$$+ \frac{2\alpha^4}{(1 + {}^1/_4\,\alpha^2\eta^2)^3}\,a_1 = \frac{\gamma^2}{3}\,\frac{8 - \alpha^2\eta^2}{(1 + {}^1/_4\,\alpha^2\eta^2)^4} - \frac{\alpha^4\beta^2\,(1 - {}^3/_4\,\alpha^2\eta^2)^2}{(1 + {}^1/_4\,\alpha^2\eta^2)^6} \tag{3.4}$$

with boundary conditions (2.14) and (2.16)

$$a_1(0) = a_1'(0) = 0, \qquad a_1(\infty) < M \tag{3.5}$$

Eq. (2.13) which defines $b_3(\eta)$ now becomes

$$b_3'' + \frac{1 + {}^5\!/_4\,\alpha^2\eta^2}{\eta\,(1 + {}^1\!/_4\,\alpha^2\eta^2)}\, b_3' + \frac{1 - {}^9\!/_2\,\alpha^2\eta^2 - {}^3\!/_{16}\,\alpha^4\eta^4}{\eta^2\,(1 + {}^1\!/_4\,\alpha^2\eta^2)^2}\, b_3 = \frac{\alpha^3\beta\gamma}{2}\,\frac{\eta\,(1 - {}^3\!/_4\,\alpha^2\eta^2)}{(1 + {}^1\!/_4\,\alpha^2\eta^2)^5} \quad (3.6)$$

with boundary conditions (2.15)

$$b_3\,(0) = b_3\,(\infty) = 0$$

If in Eqs. (3.4) and (3.6) we take

$$\xi = \frac{{}^1\!/_4\,\alpha^2\eta^2}{1 + {}^1\!/_4\,\alpha^2\eta^2} \quad (3.7)$$

as the new independent variable, we obtain after transformation

$$\xi\,(1 - \xi)^2 \frac{d^3 a_1}{d\xi^3} + (1 - \xi)\,(1 - 4\xi)\,\frac{d^2 a_1}{d\xi^2} + 6\,(1 - \xi)\,\frac{da_1}{d\xi} + 4a_1 =$$
$$= 8\,\frac{\gamma^2}{3\alpha^4}\,(2 - 3\xi) - \beta^2\,(1 - \xi)\,(1 - 4\xi)^2 \quad (3.8)$$

$$a_1\,(0) = 0, \qquad a_1'\,(0) < M, \qquad a_1\,(1) < M \quad (3.9)$$

$$\xi^2\,(1 - \xi)^2 \frac{d^2 b_3}{d\xi^2} + \xi\,(1 - \xi)\,\frac{db_3}{d\xi} - \left(\frac{1}{4} - 5\xi + 4\xi^2\right) b_3 =$$
$$= \beta\gamma\xi\,\sqrt{\xi}\,(1 - 4\xi)\,(1 - \xi)^{5/2} \quad (3.10)$$

$$b_3\,(0) = b_3\,(1) = 0$$

It will be easily seen that

$$a_1\,(\xi) = \frac{\gamma^2}{3\alpha^4}\,(5\xi - 7\xi^2) + \frac{\beta^2}{4}\,(\xi - 5\xi^2 + 4\xi^3) \quad (3.11)$$

is the partial integral of the nonhomogeneous Eq. (3.8) satisfying boundary conditions (3.9).

Solution (3.11) is the unique solution of Eq. (3.8), as the corresponding homogeneous Eq.

$$\xi\,(1 - \xi)^2 \frac{d^3 a_1}{d\xi^3} + (1 - \xi)\,(1 - 4\xi)\,\frac{d^2 a_1}{d\xi^2} + 6\,(1 - \xi)\,\frac{da_1}{d\xi} + 4a_1 = 0$$

has no solution which would satisfy boundary conditions (3.9).

In fact, the substitution of

$$a_1 = (2 - \xi - \xi^2) \int_0^{\xi} \frac{Z\,(\xi)}{(2 - \xi - \xi^2)^2}\, d\xi$$

reduces it to the second order equation

$$\xi\,(\xi - 1)\,(\xi + 2) Z'' + 2\,(\xi^2 + 3\xi - 1) Z' - 4\,(\xi + 3)\, Z = 0 \quad (3.12)$$

The integral of Eq. (3.12), bounded when $\xi = 0$, may be represented by the hypergeometric function

$$Z(\xi) = 2F\left(\frac{1 + \sqrt{17}}{2}, \frac{1 - \sqrt{17}}{2}, 1, \xi\right) + \xi F'\left(\frac{1 + \sqrt{17}}{2}, \frac{1 - \sqrt{17}}{2}, 1, \xi\right) \quad (3.13)$$

In the neighborhood of point $\xi = 1$, integral (3.13) is unbounded.

We may note that an attempt was made by Dubov in paper [3] to compute terms of the third approximation by integrating Eqs. (3.8) and (3.10) by means of expansion into infinite series. There the integral of the corresponding homogeneous equation, multiplied by an arbitrary constant, was erroneously added to the partial integral of Eq. (3.8). However, this integral will be unbounded in the neighborhood of point $\xi = 1$, and must be discarded. The solution of Eq. (3.10) given in paper [3] contains an arbitrary constant, because boundary condition (3, 15) is not satisfied.

The solution of Eq. (3, 10) which satisfies the stipulated boundary conditions is

$$b_3(\xi) = \beta\gamma\sqrt{\xi(1-\xi)}\left(-2\xi^2 + {}^7/_2\,\xi - {}^3/_2\right) + \\ + \delta\sqrt{\xi(1-\xi)}\,F\left(\frac{1+\sqrt{17}}{2},\ \frac{1-\sqrt{17}}{2},\ 2;\ \xi\right) \qquad (3.14)$$

For the determination of the arbitrary constant δ appearing in solution (3, 14) we shall stipulate the fulfilment of the last of integral conditions (2, 17).

A change to variable ξ, as defined by (3, 7), brings this integral condition to the following form

$$\int_0^1 \left(\frac{d\bar{a}}{d\xi}\,b_3 + \frac{da_0}{d\xi}\,b_2\right)\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi = 0 \qquad (3.15)$$

With parameters $\bar{a}(\eta)$, $a_0(\eta)$, $b_2(\eta)$ expressed in terms of the new variable ξ, we obtain from (3, 1) and (3, 2)

$$\bar{a} = 4\xi, \qquad a_0 = \beta\xi(2\xi - 1), \qquad b_2 = 2\gamma\sqrt{\xi}\,(1-\xi)^{3/2}$$

Substituting these expressions and the expression of $b_3(\xi)$ from (3, 14) into integral condition (3, 15), we obtain

$$\int_0^1 \Big[4\beta\gamma\sqrt{\xi(1-\xi)}\left(-2\xi^2 + {}^7/_2\,\xi - {}^3/_2\right) + \\ + 2\beta\gamma(4\xi - 1)\sqrt{\xi}\,(1-\xi)^{3/2}\Big]\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi + \\ + 4\delta\int_0^1 \xi F\left(\frac{1+\sqrt{17}}{2},\ \frac{1-\sqrt{17}}{2},\ 2;\ \xi\right) d\xi = 0$$

The first integral is equal zero. Computation of the second integral yields

$$\frac{8}{\pi}\cos\frac{\pi\sqrt{17}}{2}\,\delta = 0 \qquad \text{for} \qquad \delta = 0$$

Reverting to variable η, we derive the final Formulas for the third approximation

$$a_1(\eta) = \frac{\beta^2}{16}\,\frac{\alpha^2\eta^2\left(1 - {}^3/_4\,\alpha^2\eta^2\right)}{\left(1 + {}^1/_4\,\alpha^2\eta^2\right)^3} + \frac{\gamma^2}{24\alpha^2}\,\frac{(10 - \alpha^2\eta^2)\,\eta^2}{\left(1 + {}^1/_4\,\alpha^2\eta^2\right)^2} \qquad (3.16)$$

$$b_3(\eta) = -\frac{1}{16}\,\alpha\beta\gamma\,\frac{12 - \alpha^2\eta^2}{\left(1 + {}^1/_4\,\alpha^2\eta^2\right)^3}\,\eta \qquad (3.17)$$

From Eqs. (2, 11) we have

$$\eta c_5' = 2b_2b_3 = -\frac{1}{8}\,\alpha^2\beta\gamma^2 \times \frac{12 - \alpha^2\eta^2}{\left(1 + {}^1/_4\,\alpha^2\eta^2\right)^5}\,\eta^2$$

Taking into account conditions (2, 16) and integrating, we obtain

$$c_5 = \frac{\beta\gamma^2}{12}\,\frac{8 - \alpha^2\eta^2}{\left(1 + {}^1/_4\,\alpha^2\eta^2\right)^4} \qquad (3.18)$$

4. Computation of terms of the fourth approximation. We shall now solve Eq. (2, 10). By substituting into it the obtained values of variables a_1, c_5, a_0 and changing over to variable ξ defined by Formula (3, 7), we derive the nonhomogeneous linear equation of the third order which defines function a_2

$$\xi(1-\xi)^2\frac{d^3a_2}{d\xi^3}+(1-\xi)(1-4\xi)\frac{d^2a_2}{d\xi^2}+8(1-\xi)\frac{da_2}{d\xi}+8a_2=$$
$$=\tfrac{1}{8}\beta^3(192\,\xi^4-416\,\xi^3+288\,\xi^2-69\,\xi+5)+$$
$$+\tfrac{1}{6}\beta\gamma^2\alpha^{-4}(-196\,\xi^3+252\,\xi^2-33\,\xi-15) \qquad (4.1)$$

with boundary conditions

$$a_2(0)=0, \qquad a_2'(0)<M, \qquad a_2(1)<M \qquad (4.2)$$

A solution of Eq. (4.1) which satisfies conditions (4.2) is

$$a_2=\tfrac{1}{8}\beta^3\;[4\xi^4-8\xi^3+{}^{11}/_3\xi^2+{}^{7}/_{12}\xi+(1-\xi)\ln\;(1-\zeta)]- \qquad (4.3)$$
$$-\tfrac{1}{12}\beta\gamma^2\alpha^{-4}\cdot(28\xi^3-45\xi^2+15\xi)+\delta\;[13\xi-10\xi^2+12\,(1-\xi)\ln\,(1-\xi)]$$

Here, δ is an arbitrary constant. Using Eq. (2.4) we obtain the expression of the axial component of the velocity vector in terms of variable ξ in the fourth approximation

$$u\,(x,\;\xi)=\tfrac{1}{2}\alpha^2\,(1-\xi^2)[a'\,(\xi)x^{-1}+a_0'\,(\xi)x^{-2}+a_1'\,(\xi)\;x^{-3}+a_2'\;(\xi)x^{-4}] \qquad (4.4)$$

Here

$$a'\,(\xi)=4,\; a_0'(\xi)=\beta\,(4\xi-1)$$
$$a_1'\,(\xi)=\tfrac{1}{4}\beta^2\,(1-10\xi+12\xi^2)+\tfrac{1}{3}\gamma^2\alpha^{-4}(5-14\xi)$$
$$a_2'\,(\xi)=\tfrac{1}{8}\beta^3\;[16\xi^3-24\xi^2+{}^{22}/_3\,\xi-{}^{5}/_{12}-\ln\,(1-\xi)]-$$
$$-\tfrac{1}{4}\,\beta\gamma^2\alpha^{-2}\,(28\xi^2-30\xi+5)+\delta\;[-20\xi+1-\ln\;(1-\xi)]$$

With $\xi=0$ we obtain from this the axial velocity distribution on the stream axis

$$u\,(x,\,0)=\frac{\alpha^2}{2}\left[\frac{4}{x}-\frac{\beta}{x^2}+\frac{1}{x^3}\left(\frac{\beta^2}{4}+\frac{5\gamma^2}{3\alpha^4}\right)-\frac{1}{x^4}\left(\frac{5}{96}\beta^3+\frac{5\beta\gamma^2}{4\alpha^4}-\delta\right)\right] \quad (4.5)$$

The reverse flow, which is the result of the stream twist vanishes at point x_0 where $u(x_0)=0$.

For the determination of this point we obtain from (4.5) a third power equation with respect to x_0

$$4x_0^3-\beta x_0^2+\left(\frac{\beta^2}{4}+\frac{5\gamma^2}{3\alpha^4}\right)x_0-\left(\frac{5}{96}\beta^3-\delta+\frac{5\beta\gamma^2}{4\alpha^4}\right)=0 \qquad (4.6)$$

Under conditions of no-twist of the stream the area of reverse flow must be absent. Hence, with $\gamma=0$, Eq. (4.6) must have a root $x_0=0$. We then have $\delta={}^5/_{96}\beta^3$, and (4.6) becomes

$$4x_0^3-\beta x_0^2+\left(\frac{\beta^2}{4}+\frac{5\gamma^2}{3\alpha^4}\right)x_0-\frac{5\beta\gamma^2}{4\alpha^4}=0 \qquad (4.7)$$

Eq. (4.7) has one positive real root. The asymptotic expansion presented in this paper is not valid in the case of $x<x_0$, as then $u<0$. In this case we have a zone of reverse flows to which the boundary layer theory is not applicable. It follows from Eq. (4.7) that x_0 is expressed by

$$x_0=\beta f\left(\frac{\gamma^2}{\alpha^4\beta^2}\right), \qquad \beta=\frac{M_0}{2\pi\mu}.$$

Here, M_0 is the initial mass flow per second [2]. In this way two twisted streams having the same ratio $\gamma^2/\alpha^4\beta^2$ will be similar. Expressing α, β, γ in terms of initial impulses and of the kinetic moment in accordance with Formulas (3.3), we obtain the following criterion for the similarity of twisted streams

$$S=\frac{\rho L_0^2}{K_0M_0^2}$$

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ON A LINEAR INSTABILITY OF A PLANE PARALLEL COUETTE FLOW OF VISCOELASTIC FLUID

(O LINEINOI NEUSTOICHIVOSTI PLOSKOPARALLEL'NOGO TECHENIIA KUETTA UPRUGO-VIAZKOI ZHIDKOSTI)

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Numerous investigations have shown that majority of real fluids cannot be described in terms of a constant viscosity (Newtonian) model. There exist various effects connected with elastic properties of these fluids, with the dependence of parameters on shear velocity etc.

It is these phenomena, inherently related to the nonnewtonian behavior of melts and solutions of polymers, that cause irregularities in their flow patterns and give rise to so called "elastic turbulence" [1 to 3].

All the flow irregularities exhibit a common characteristic feature, namely they appear at very small Reynolds numbers (they are very high viscosity fluids), when the usual hydrodynamic instability and turbulence cannot take place.

Assumption of the "elastic" character of this phenomenon is well supported by experimental data available, and several authors [4 to 6] use the critical value of a dimensionless parameter $\Gamma = \theta VL^{-1} = \eta G^{-1}VL^{-1}$, characterizing reversible elastic deformation of fluid, as the criterion of its appearance. Here η is the viscosity, G is the shear modulus, $\theta = \eta G^{-1}$ is the time of relaxation while V and L are characteristic velocity and linear dimension, respectively.

When the accumulated elastic deformation exceeds some critical value (of the order of 7), then the phenomenon described above takes place, and we can use this as a basis for another assumption. Just as the inertial forces in a viscous fluid, the elastic forces act, in viscoelastic fluids as an additional destabilizing factor (the connection between the elastic terms and additional nonlinearity in equations will be seen later on the model used). This in turn, leads to consideration of the possibility of a special "elastic"